TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 304, Number 1, November 1987

CONTRIBUTIONS TO THE THEORY OF SET VALUED FUNCTIONS AND SET VALUED MEASURES

NIKOLAOS S. PAPAGEORGIOU

ABSTRACT. Measurable multifunctions and multimeasures with values in a Banach space are studied. We start by proving a variation of the known Dunford theorem for weak compactness in $L^1(X)$. With a similar technique we prove that the range of certain vector valued integrals that appear in applications is w-compact and convex. Also we obtain Dunford-Pettis type theorems for sequences of integrably bounded multifunctions. Some pointwise w-compactness theorems are also obtained for certain families of measurable multifunctions. Then we prove a representation theorem for additive, set valued operators defined on $L^1(X)$. Finally, in the last section, a detailed study of transition multimeasures is conducted and several representation theorems are proved.

1. Introduction. In this paper we continue the work started in [**31**], where we extended some well-known functional analytic results to set valued operators and used techniques from the general theory of multifunctions to obtain a new result about weak compactness in the Lebesgue-Bochner space $L^1(X)$. This effort is continued in this paper, where we have a new weak compactness result for $L^1(X)$ that complements some earlier results of Brooks and Dinculeanu [**2**] and we present a Dunford-Pettis type theorem for sequences of integrably bounded multifunctions. We also have a new representation theorem for additive set valued operators, some pointwise weak compactness results for multifunctions with a weakly compact set of Bochner integrable selectors, another weak compactness theorem for the space $L^1(X)$, and we determine various properties of transition multimeasures. Our proofs are based on results from the general theory of measurable multifunctions and multimeasures which can be found in the works of Castaing and Valadier [**3**], Costé [**5, 6**], Godet-Thobie [**14, 15**], Hiai [**16, 17**], Hiai and Umegaki [**18**], Pallu de la Barrière [**30**], and Saint-Beuve [**29, 40, 41**]. Also for the relevant background on vector measures, the Radon-Nikodým property (R.N.P.), and Bochner or Pettis integration the reader is referred to Diestel and Uhl [**9**]. Our work was partially motivated by problems in mathematical economics, control theory, and optimization, where the results and techniques developed in this paper can have useful applications.

Received by the editors June 27, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 28A45, 46G10, 46E30.
Research supported by NSF Grant DMS-8602313.

2. Preliminaries. In this section we establish our notation and terminology and also we present some basic facts from the theory of multifunctions that we will need in the sequel.

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X: \text{nonempty, closed, (convex)}\},$$

$$P_{(w)k(c)}(X) = \{A \subseteq X: \text{nonempty, } (w\text{-})\text{compact, (convex)}\}.$$

Also for $A \in 2^X \setminus \{\varnothing\}$, we define $|A| = \sup\{\|\alpha\|: \alpha \in A\}$ (the "norm" of the set A), $\sigma(x^*, A) = \sup\{(x^*, \alpha): \alpha \in A\}$, $x^* \in X^*$ (support function of A), and $d(x, A) = \inf\{\|x - \alpha\|: \alpha \in A\}$ (the distance function from A).

A multifunction $F: \Omega \to P_f(X)$ is said to be measurable if one of the following two equivalent conditions is satisfied:

(i) for all $x \in X$, $\omega \to d(x, F(\omega))$ is measurable;

(ii) there exist $f_n: \Omega \to X$ measurable functions s.t.

$$F(\omega) = \text{cl}\{f_n(\omega)\}_{n \geqslant 1} \quad \text{for all } \omega \in \Omega$$

(Castaing's representation).

If in addition there exists a complete, σ-finite measure $\mu(\cdot)$ on Σ, then (i) and (ii) above are equivalent to

(iii) $\text{Gr} F = \{(\omega, x) \in \Omega \times X: x \in F(\omega)\} \in \Sigma \times B(X)$, where $B(X)$ is the Borel σ-field of X (graph measurability).

By S_F^p $(1 \leqslant p \leqslant \infty)$ we will denote the set of selectors of a measurable multifunction $F: \Omega \to P_f(X)$, which belongs to the Lebesgue-Bochner space $L^p(X)$. This set (which may be empty) is closed in $L^p(X)$. It is easy to see that S_F^p is nonempty if and only if $\inf\{\|x\|: x \in F(\omega)\} \in L^p$. We will say that $F(\cdot)$ is integrably bounded if $\omega \to |F(\omega)|$ belongs in L^1. Using the Kuratowski-Ryll-Nardzewski measurable selection theorem, we can easily see that for an integrably bounded multifunction $F(\cdot)$, $S_F^1 \neq \varnothing$. Having this set we can define a set valued integral for $F(\cdot)$ as

$$\int_\Omega F(\omega)\, d\mu(\omega) = \left\{\int_\Omega f(\omega)\, d\mu(\omega): f \in S_F^1\right\}.$$

The vector valued integrals of the right-hand side are defined in the sense of Bochner.

A set $K \subseteq L^1(X)$ is said to be decomposable (or "convex with respect to switching") if for all $f, g \in K$ and all $A \in \Sigma$ we have $\chi_A f + \chi_{A^c} g \in K$. Decomposability plays an important role in control theory and optimization.

If $\Sigma_0 \subseteq \Sigma$ is a sub-σ-field and $F: \Omega \to P_f(X)$ is integrably bounded, then from Hiai and Umegaki **[18]** we know that there exists a unique multifunction $E^{\Sigma_0}F$: $\Omega \to P_f(X)$ which is Σ_0-measurable, integrably bounded and $S^1_{E^{\Sigma_0}F} = \text{cl}\{E^{\Sigma_0}f$: $f \in S_F^1\}$, the closure taken in the norm topology of $L^1(X, \Sigma_0)$.

Now let X be any Banach space. A multimeasure is a map $M: \Sigma \to 2^X \setminus \{\varnothing\}$ s.t. $M(\varnothing) = \{0\}$ and for $\{A_n\}_{n \geqslant 1} \subseteq \Sigma$ pairwise disjoint we have

$$M\left(\bigcup_{n \geqslant 1} A_n\right) = \sum_{n \geqslant 1} M(A_n).$$

Depending on how we interpret the sum in the right-hand side, we have different types of multimeasures. Here we present the two basic ones that we will be using in this work.

(a) $M(\cdot)$ is a "multimeasure" (or "strong multimeasure") if and only if $\Sigma_{n \geqslant 1} M(A_n) = \{x \in X\colon x = \Sigma_{n \geqslant 1} x_n$ (unconditionally convergent), $x_n \in M(A_n)\}$.

(b) If the values of $M(\cdot)$ are closed, we define $M(\cdot)$ as a "weak multimeasure" if and only if for every $x^* \in X^*$, $A \to \sigma(x^*, M(A))$ is a signed measure.

We know (see Godet-Thobie [**14**, Proposition 3] and Pallu de la Barriére [**30**, p. 7–07]), that if $M(\cdot)$ is $P_{wkc}(X)$-valued, then the above two definitions are equivalent. This fact can be viewed as the multivalued analogue of the Orlicz-Pettis theorem (see Diestel and Uhl [**9**]). In fact in this case

$$\sum_{n=1}^{m} M(A_n) \xrightarrow{h} M\Big(\bigcup_{n \geqslant 1} A_n\Big) \quad \text{as } m \to \infty,$$

where $h(\cdot, \cdot)$ denotes the Hausdorff metric on $P_{wkc}(X)$.

If $M(\cdot)$ is a multimeasure and $A \in \Sigma$, then we define

$$|M|(A) = \sup_{\pi} \sum_{\pi} |M(A_n)|,$$

where the supremum is taken over all finite measurable partitions $\pi = \{A_1, \dots, A_n\}$ of A. If $|M|\Omega < \infty$ then $M(\cdot)$ is said to be of bounded variation. It is easy to see that $|M|(\cdot)$ is a positive measure. Also $A \in \Sigma$ is said to be an atom for $M(\cdot)$ if $M(A) \neq \{0\}$ and $M(B) = \{0\}$ or $M(A \setminus B) = \{0\}$ for all $B \subseteq A$, $B \in \Sigma$. Clearly $A \in \Sigma$ is an atom for $M(\cdot)$ if and only if it is an atom for $|M|(\cdot)$. If $M(\cdot)$ has no atoms it is called nonatomic. Finally $M(\cdot)$ is said to be $\mu(\cdot)$-continuous if $\mu(A) = 0$ implies $M(A) = \{0\}$. Again $M(\cdot)$ is μ-continuous if and only if $|M|(\cdot)$ is. A vector measure $m\colon \Sigma \to X$ s.t. $m(A) \in M(A)$ for all $A \in \Sigma$ is said to be a measure selector of $M(\cdot)$. The set of all measure selectors of $M(\cdot)$ is denoted by S_M.

Now let (Ω, Σ), (T, Z) be measurable spaces and X be a separable Banach space. A map $L\colon \Omega \times Z \to P_f(X)$ is said to be a transition multimeasure (resp. a weak transition multimeasure) if:

(1) for all $A \in Z$, $\omega \to L(\omega, A)$ is a measurable miltifunction,

(2) for all $\omega \in \Omega$, $A \to L(\omega, A)$ is a multimeasure (resp. a weak multimeasure).

A "selector transition measure" or simply a "transition selector" is a map $m\colon \Omega \times Z \to X$ s.t.

(1) for all $A \in Z$, $\omega \to m(\omega, A)$ is Σ-measurable,

(2) for all $\omega \in \Omega$, $A \to m(\omega, A)$ is a vector measure,

(3) for all $\omega \in \Omega$ and all $A \in Z$, $m(\omega, A) \in L(\omega, A)$.

The set of all transition selectors of $L(\cdot, \cdot)$ will be denoted by TS_L.

Finally recall (see Diestel and Uhl [**9**, p. 74]) that $K \subseteq L^1(X)$ is uniformly integrable if $\lim_{\mu(A) \to 0} \int_A \|f(\omega)\| \, d\mu(\omega) = 0$ uniformly in $f(\cdot) \in K$.

Closing this section we would like to point out that several of the results in this paper can be extended to separable Fréchet spaces. But in order to have a uniform exposition we have decided to present everything within the framework of Banach spaces.

3. Weak compactness in $L^1(X)$. We will start with a variation of the well-known Dunford compactness theorem (see Diestel and Uhl [**9**, p. 101]). Note that we do not require X to have the R.N.P. and instead we have the weaker hypothesis (3) and separability. However we require that our set $K \subseteq L^1(X)$ is convex. Our result is also related to Theorem 1 of Brooks and Dinculeanu [**2**]. Furthermore our proof is different from those in [**2** and **9**] and is based on the theory of multimeasures. Our theorem was motivated by a problem in mathematical economics.

THEOREM 3.1. *Assume that* (Ω, Σ, μ) *is a positive, finite measure space and* X *a separable Banach space. If* $K \subseteq L^1(X)$ *is closed, convex, bounded, and*

(1) *K is uniformly integrable,*

(2) *for all* $A \in K$, $K(A) = \{ \int_A f(\omega)\, d\mu(\omega) \colon f \in K \}$ *is w-compact in* X,

(3) *for all vector measures* $m\colon \Sigma \to X$ *s.t.* $m(A) \in K(A)$ *for all* $A \in \Sigma$ *there exists* $g \in L^1(X)$ *s.t.* $m(A) = \int_A g(\omega)\, d\mu(\omega)$,

(4) $[L^1(X)]^* = L^\infty(X^*)$,

then K is weakly compact in $L^1(X)$.

PROOF. Consider the map $M\colon \Sigma \to 2^X \setminus \{\varnothing\}$ defined by $M(A) = \{ \int_A f(\omega)\, d\mu(\omega)\colon f \in K \}$. Because of hypothesis (2) and the convexity of K, we deduce that for all $A \in \Sigma$, $M(A) \in P_{wkc}(X)$. Next fix $x^* \in X^*$. We have

$$\sigma(x^*, M(A)) = \sup_{f \in K} \left(x^*, \int_A f(\omega)\, d\mu(\omega) \right) = \sup_{f \in K} \int_A (x^*, f(\omega))\, d\mu(\omega).$$

Consider the set $K_{x^*} \subseteq L^1$ defined by

$$K_{x^*} = \{ (x^*, f(\cdot)) \in L^1 \colon f \in K \}.$$

Because of hypothesis (1), K_{x^*} is uniformly integrable and also bounded. So from the classical Dunford-Pettis criterion $\overline{K_{x^*}}^w$ is w-compact in L^1. Hence there exists $u(\cdot) \in \overline{K_{x^*}}^w$ s.t. $\sigma(x^*, M(A)) = \int_A u(\omega)\, d\mu(\omega) \Rightarrow A \to \sigma(x^*, M(A))$ is a signed measure $\Rightarrow M(\cdot)$ is a weak multimeasure on Σ (in fact a multimeasure since it is $P_{wkc}(X)$-valued).

Next imbed K into $[L^1(X)]^{**}$ and consider $\overline{K}^{w^*}$. Because of Alaoglu's theorem $\overline{K}^{w^*}$ is w^*-compact. To prove our theorem it suffices to show that for every $v \in \overline{K}^{w^*}$, we have $v \in L^1(X)$. For every $A \in \Sigma$ and for every $x^* \in X^*$ let $(x^*, m(A)) = \langle \chi_A x^*, v \rangle$. Take a net $\{f_b\}_{b \in B} \subseteq K$ (B = directed set) s.t. $f_b \xrightarrow{w^*} v$. Then $(x^*, \int_A f_b(\omega)\, d\mu(\omega)) = (x^*, m_b(A)) \to (x^*, m(A))$. Note that $m_b \in S_M$, $b \in B$; from the first part of the proof we know that $M(\cdot)$ is a $P_{wkc}(X)$-valued multimeasure and from Theorem 1 of Godet-Thobie [**14**] we know then that S_M is compact for the topology of pointwise w-convergence. Thus we get that $m \in S_M$. Then using hypothesis (3) we can find $g \in L^1(X)$ s.t.

$$(x^*, m(A)) = \int_A (x^*, g(\omega))\, d\mu(\omega) \Rightarrow \langle \chi_A x^*, v \rangle = \int_A (x^*, g(\omega))\, d\mu(\omega)$$

and so $\langle s, v \rangle = \int_A (s(\omega), g(\omega))\, d\mu(\omega)$ for all $s(\cdot) \in L^\infty(X^*)$ countably valued simple functions. Recalling that those functions are dense in $L^\infty(X^*)$ (see Diestel and Uhl [**9**, p. 42]), we conclude that for all $w \in L^\infty(X^*)$ we have

$$\langle w, v \rangle = \int_\Omega (w(\omega), v(\omega))\, d\mu(\omega) \Rightarrow v \in L^1(X) \Rightarrow K = \overline{K}^{w^*} \Rightarrow K$$

is w-compact in $L^1(X)$. Q.E.D.

REMARKS. From Theorem 1 of Diestel and Uhl [9, p. 98], we know that hypothesis (4) is equivalent to saying that X^* has the R.N.P. with respect to μ and if μ is nonatomic, then this is equivalent to saying that X^* is separable. However note that hypothesis (3) is weaker than saying that X has the R.N.P.

Using a similar technique involving multimeasures, we can have the following result on the range of certain vector valued integrals. Our result extends those of Cesari [4] and is useful in control theory and optimization.

THEOREM 3.2. *Assume that* (Ω, Σ, μ) *is a nonatomic, positive, finite measure space and* X *is a Banach space with the R.N.P. If* $K \subseteq L^1(X)$ *is closed, convex, bounded, and*

(1) K *is uniformly integrable,*

(2) *for all* $A \in \Sigma$, $K(A) = \{\int_A f(\omega)\, d\mu(\omega) \colon f \in K\}$ *is w-compact in* X, *then* $\overline{\bigcup_{A \in \Sigma} K(A)}$ *is w-compact and convex in* X.

PROOF. Again consider $M\colon \Sigma \to 2^X \setminus \{\varnothing\}$ defined by $M(A) = \{\int_A f(\omega)\, d\mu(\omega)\colon f \in K\}$. As we have already seen in the proof of Theorem 3.1, $M(\cdot)$ is a $P_{wkc}(X)$-valued multimeasure which clearly is of bounded variation. Also $M \ll \mu$ and since $\mu(\cdot)$ is nonatomic, so is $M(\cdot)$. Invoking Theorem 5 of Costé [6] we deduce that $\overline{\bigcup_{A \in \Sigma} K(A)} = \overline{\bigcup_{A \in \Sigma} M(A)}$ is convex. Next we will show that it is also w-compact. Let $x^* \in X^*$ and let Ω_+, Ω_- be a Hahn decomposition for the signed measure $A \to \sigma(x^*, M(A))$. Then we have

$$\sigma\left(x^*, \overline{\bigcup_{A \in \Sigma} M(A)}\right) = \sup_{A \in \Sigma} \sigma\big(x^*, M(A)\big)$$

$$= \sup_{A \in \Sigma} \sigma\big(x^*, M(\Omega_+ \cap A)\big) = \sigma\big(x^*, M(\Omega_+)\big).$$

But $M(\Omega_+) \in P_{wkc}(X)$, so we can find $\hat{x} \in M(\Omega_+)$, depending on x^*, s.t. $\sigma(x^*, M(\Omega_+)) = (x^*, \hat{x}) \Rightarrow \sigma(x^*, \overline{\bigcup_{A \in \Sigma} M(A)}) = (x^*, \hat{x})$. Invoking James' theorem we conclude that $\overline{\bigcup_{A \in \Sigma} M(A)} = \overline{\bigcup_{A \in \Sigma} K(A)}$ is w-compact in X. Q.E.D.

REMARK. It is clear from the above proof that the nonatomicity of $\mu(\cdot)$ and the R.N.P. on X were needed only for the convexity part of the conclusion (see Theorem 5 of Costé [6]).

In fact, in the light of the previous remark, an interesting byproduct of the proof of Theorem 3.2 is the following result which extends the well-known theorem of Bartle-Dunford-Schwartz (see [9, p. 14]) to multimeasures.

THEOREM 3.3. *Assume that* (Ω, Σ) *is a measurable space and* X *a Banach space. If* $M\colon \Sigma \to P_{wk}(X)$ *is a weak multimeasure then* $\overline{\bigcup_{A \in \Sigma} M(A)}$ *is w-compact in* X.

4. Weak compactness in the space of integrable multifunctions. In this section we present some Dunford-Pettis type compactness theorems for $P_{wkc}(X)$-valued, integrably bounded multifunctions.

If $\{A_n, A\}_{n \geq 1} \subseteq 2^X \setminus \{\varnothing\}$, we say that the A_n's converge weakly to A (denoted by $A_n \xrightarrow{w} A$) if and only if for all $x^* \in X^*$, $\sigma(x^*, A_n) \to \sigma(x^*, A)$.

THEOREM 4.1. *Assume that* (Ω, Σ, μ) *is a positive, finite, complete measure space and* X *a separable Banach space. If* $F_n\colon \Omega \to P_{wkc}(X)$, $n \geqslant 1$, *are integrably bounded and* $F_n(\omega) \subseteq W$ μ-*a.e., where* $W \in P_{wkc}(X)$ *then there exists* $F\colon \Omega \to P_{wkc}(X)$ *integrably bounded and a subsequence* $\{F_{n_k} = F_k\}_{k\geqslant 1} \subseteq \{F_n\}_{n\geqslant 1}$ *s.t. for all* $A \in \Sigma$ *we have* $\int_A F_k(\omega)\,d\mu(\omega) \xrightarrow{w} \int_A F(\omega)\,d\mu(\omega)$.

PROOF. For each $A \in \Sigma$, let $M_n(A) = \int_A F_n(\omega)\,d\mu(\omega)$, $n \geqslant 1$. Using Theorem 2.2 of Hiai and Umegaki [**18**], for all $x^* \in X^*$ we have $\sigma(x^*, M(A_n)) = \int_A \sigma(x^*, F_n(\omega))\,d\mu(\omega)$. Fix $x^* \in X^*$. Using the Dunford-Pettis compactness criterion we know that there exists

$$\left\{\sigma\left(x^*, F_{n_k}(\cdot)\right) = \sigma\left(x^*, F_k(\cdot)\right)\right\}_{k\geqslant 1} \subseteq \left\{\sigma\left(x^*, F_n(\cdot)\right)\right\}_{n\geqslant 1}$$

and $h(\cdot, x^*) \in L^1$ s.t. for all $A \in \Sigma$ we have

$$\int_A \sigma\left(x^*, F_k(\omega)\right) d\mu(\omega) \to \int_A h(\omega, x^*)\,d\mu(\omega) \Rightarrow \sigma\left(x^*, M_k(A)\right)$$

$$\to \varphi(A, x^*) = \int_A h(\omega, x^*)\,d\mu(\omega).$$

From Nikodým's theorem we know that $A \to \varphi(A, x^*)$ is a signed measure on R. Also note that for all $A \in \Sigma$ we have

$$M_k(A) = \int_A F_k(\omega)\,d\mu(\omega) \subseteq \int_A W d\mu(\omega) = \mu(A)W.$$

By considering, if necessary, $\overline{\text{conv}}[W \cup (-W)]$ which is still in $P_{wkc}(X)$ (Krein-Smulian theorem, see [**9**, p. 51]), we may assume that W is symmetric. Then

$$\left|\sigma\left(x^*, M_k(A)\right)\right| \leqslant \mu(A)\sigma(x^*, W) \Rightarrow \left|\varphi(A, x^*)\right| \leqslant \mu(A)\sigma(x^*, W).$$

Since $W \in P_{wkc}(X)$, $\sigma(\cdot, W)$ is continuous for the Mackey topology $m(X^*, X)$ (see for example Theorem 6.3.9 of Laurent [**26**]). So from the last inequality we get that $x^* \to \varphi(A, x^*)$ is $m(X^*, X)$-continuous. Hence combining Theorem II-16 of Castaing and Valadier [**3**], with Theorem 6.3.9 of Laurent [**26**], we deduce that there exists $M(A) \in P_{wkc}(X)$ s.t. $\varphi(A, x^*) = \sigma(x^*, M(A))$. Clearly then the map M: $\Sigma \to P_{wkc}(X)$ is a weak multimeasure (in fact a multimeasure) and $M(A) \subseteq \mu(A)W$. So we can apply Theorem 1 of Godet-Thobie [**15**] and get $F\colon \Omega \to P_{wkc}(X)$ integrably bounded s.t. for all $A \in \Sigma$ and all $x^* \in X^*$ we have $\sigma(x^*, M(A)) = \int_A \sigma(x^*, F(\omega))\,d\mu(\omega)$. Recalling that $\int_A \sigma(x^*, F_k(\omega))\,d\mu(\omega) = \sigma(x^*, \int_A F_k)$ and $\int_A \sigma(x^*, F(\omega))\,d\mu(\omega) = \sigma(x^*, \int_A F)$, we conclude that $\int_A F_k \xrightarrow{w} \int_A F$ for all $A \in \Sigma$. Q.E.D.

We can have an alternative version of the previous theorem.

THEOREM 4.2. *Assume that* (Ω, Σ, μ) *is a positive, finite, complete measure space and* X *a separable Banach space with* X^* *separable too. If* $F_n\colon \Omega \to P_{wkc}(X)$, $n \geqslant 1$, *are integrably bounded and*

(1) $\{|F_n(\cdot)|\}_{n\geqslant 1}$ *is uniformly integrable and* L^1-*bounded*,

(2) *for all* $A \in \Sigma$, $K(A) = w\text{-cl}[\bigcup_{n\geqslant 1} \int_A F_n(\omega)\,d\mu(\omega)]$ *is w-compact in* X,

(3) *every vector measure* $m\colon \Sigma \to X$ *s.t.* $m(A) \in \overline{\text{conv}}\, K(A)$ *for all* $A \in \Sigma$ *admits a density in* $L^1(X)$,
then there exists $F\colon \Omega \to P_{wkc}(X)$ *integrably bounded and a subsequence* $\{F_{n_k} = F_k\}_{k \geqslant 1} \subseteq \{F_n\}_{n \geqslant 1}$ *s.t. for all* $A \in \Sigma$

$$\int_A F_k(\omega)\, d\mu(\omega) \xrightarrow{w} \int_A F(\omega)\, d\mu(\omega).$$

PROOF. Working as in the proof of Theorem 4.1 we produce $\varphi(A, x^*)$ s.t. $\varphi(\cdot, x^*)$ is a signed measure. Also note that $|\varphi(A, x^*)| \leqslant \sigma(x^*, \hat{K}(A))$, where $\hat{K}(A) = \overline{\text{conv}}[K(A) \cup (-K(A))] \in P_{wkc}(X)$. Again $x^* \to \sigma(x^*, \hat{K}(A))$ is $m(X^*, X)$-continuous and hence so is $x^* \to \varphi(A, x^*)$. Thus we can find $M(A) \in P_{wkc}(X)$ s.t. $\varphi(A, x^*) = \sigma(x^*, M(A)) \Rightarrow M(\cdot)$ is a multimeasure. Note that $M(\cdot)$ is of bounded variation and μ-continuous. So Theorem 3 of Costé [5] tells us that there exists $F\colon \Omega \to P_{wkc}(X)$ integrably bounded s.t. for all $A \in \Sigma$ we have $M(A) = \int_A F(\omega)\, d\mu(\omega) \Rightarrow \sigma(x^*, M(A)) = \sigma(x^*, \int_A F)$. Therefore $\sigma(x^*, \int_A F_k) \to \sigma(x^*, \int_A F) \Rightarrow \int_A F_k \xrightarrow{w} \int_A F$ for all $A \in \Sigma$. Q.E.D.

REMARK. If X has the R.N.P., then hypothesis (3) is automatically satisfied.

When X is finite dimensional, then we can say more. Recall that by $h(\cdot, \cdot)$ we denote the Hausdorff (generalized) metric on $P_f(X)$.

THEOREM 4.3. *Assume that* (Ω, Σ, μ) *is a positive, finite, complete measure space and* X *a finite dimensional Banach space. If* $F_n\colon \Omega \to P_{fc}(X)$, $n \geqslant 1$, *are measurable multifunctions s.t.* $\{|F_n(\cdot)|\}_{n \geqslant 1}$ *is uniformly integrable, then there exists* $F\colon \Omega \to P_{fc}(X)$ *integrably bounded and a subsequence* $\{F_{n_k} = F_k\}_{k \geqslant 1} \subseteq \{F_n\}_{n \geqslant 1}$ *s.t. for all* $A \in \Sigma$

$$\int_A F_k(\omega)\, d\mu(\omega) \xrightarrow{h} \int_A F(\omega)\, d\mu(\omega).$$

PROOF. Because of the finite dimensionality of X, hypotheses (2) and (3) of Theorem 4.2 are automatically satisfied. So applying that theorem we get

$$\int_A F_k(\omega)\, d\mu(\omega) \xrightarrow{w} \int_A F(\omega)\, d\mu(\omega) \Rightarrow \sigma\left(x^*, \int_A F_k\right) \to \sigma\left(x^*, \int_A F\right).$$

Using Corollary 2C of Salinetti and Wets [42] we get that $\sigma(\cdot, \int_A F_k) \xrightarrow{T} \sigma(\cdot, \int_A F)$ which by Theorem 3.1 of Mosco [28] means that $\int_A F_k(\omega)\, d\mu(\omega) \to \int_A F(\omega)\, d\mu(\omega)$ in the Kuratowski sense (see [43]). Finally applying Corollary 3A of Salinetti and Wets [43], we get that $\int_A F_k(\omega)\, d\mu(\omega) \xrightarrow{h} \int_A F(\omega)\, d\mu(\omega)$. Q.E.D.

5. Pointwise weak compactness of a multifunction. In this section, starting from the set of integrable selectors of a multifunction, we extract information about its pointwise properties and the pointwise properties of its conditional expectation. Such results are useful in mathematical economics, because, roughly speaking, what they mean is that properties of the set of all feasible consumption allocations of the totality of agents are transferred to the consumption correspondence of each individual agent.

Let X be a Banach space and let $k > 0$. Following Rosenthal [**39**] we say that a bounded sequence $\{x_n\}_{n \geqslant 1} \subseteq X$ is k-equivalent to the canonical basis of l^1 if for all $n \geqslant 1$ and for all $b_1, \ldots, b_n \in R$ we have

$$\sum_{j \leqslant n} |b_j| \leqslant k \left\| \sum_{j \leqslant n} b_j x_j \right\|.$$

We say that $\{x_n\}_{n \geqslant 1}$ is an l^1-sequence if there exists $k > 0$ s.t. $\{x_n\}_{n \geqslant 1}$ is k-equivalent to the l^1-basis. Note that if $\{x_n\}_{n \geqslant 1}$ is an l^1-sequence, $\overline{\text{span}}\{x_n\}_{n \geqslant 1}$ is isomorphic to l^1 and so $l^1 \hookrightarrow X$.

From Rosenthal [**39**] we have the following remarkable theorem.

THEOREM 5.1 [**39**]. *Assume that X is a Banach space. From every bounded sequence $\{x_n\}_{n \geqslant 1}$ in X we can extract a subsequence $\{x_{n_k} = x_k\}_{k \geqslant 1}$ which has one and only one of the following two properties.*

(1) *$\{x_k\}_{k \geqslant 1}$ is an l^1-sequence.*

(2) *$\{x_k\}_{k \geqslant 1}$ is weakly Cauchy.*

We start with a lemma that we will need in the sequel.

LEMMA I. *Assume that (Ω, Σ, μ) is a positive, finite measure space and X a weakly sequentially complete Banach space. A subset K of $L^1(X)$ is relatively w-compact if and only if K is bounded and has no l^1-sequence.*

PROOF. *K is bounded with no l^1-sequence.* Let $\{f_n\}_{n \geqslant 1} \subseteq K$. From Theorem 5.1 we know that there exists $\{f_{n_k} = f_k\}_{k \geqslant 1} \subseteq \{f_n\}_{n \geqslant 1}$ weakly Cauchy in $L^1(X)$. Furthermore from Talagrand [**44**] we know that since X is weakly sequentially complete, so is $L^1(X)$. Therefore $f_k \xrightarrow{w} f \in L^1(X)$. Invoking the Eberlein-Smulian theorem (see Dunford and Schwartz [**12**, p. 430]) we conclude that K is relatively w-compact.

K is a relatively w-compact subset of $L^1(X)$. Clearly then K is bounded. Also from the Eberlein-Smulian theorem and Theorem 5.1 we conclude that K has no l^1-sequence. Q.E.D.

This allows us to have the following theorem.

THEOREM 5.2. *Assume that (Ω, Σ, μ) is a positive, finite measure space and X a weakly sequentially complete, separable Banach space. If K is a decomposable, w-compact convex subset of $L^1(X)$ then there exists $F\colon \Omega \to P_{wkc}(X)$ integrably bounded s.t. $K = S_F^1$.*

PROOF. From Theorems 3.1 and 3.2 and Corollary 1.6 of Hiai and Umegaki [**18**] we know that there exists $F\colon \Omega \to P_{fc}(X)$ integrably bounded s.t. $K = S_F^1$. Furthermore from Castaing's representation (see §2) we have $F(\omega) = \text{cl}\{f_n(\omega)\}_{n \geqslant 1} \in P_{fc}(X)$ with $f_n \in S_F^1$, $n \geqslant 1$. From Lemma I we know that S_F^1 has no l^1-sequence. Hence from Klei [**24**], we know that $\{f_n(\omega)\}_{n \geqslant 1}$ has no l^1-sequence for all $\omega \in \Omega \setminus N$, $\mu(N) = 0$. So Theorem 5.1 tells us that $\{f_n(\omega)\}_{n \geqslant 1}$ has a weakly Cauchy subsequence for $\omega \in \Omega \setminus N$, $\mu(N) = 0$. Then from the weak sequential completeness of X and the Eberlein-Smulian theorem, we deduce that $\text{cl}\{f_n(\omega)\}_{n \geqslant 1} \in P_{wkc}(X) \Rightarrow F(\omega) \in P_{wkc}(X)$ for all $\omega \in \Omega \setminus N$, $\mu(N) = 0$. By redefining $F(\cdot)$ on N, we get the conclusion of the theorem. Q.E.D.

An immediate, important consequence of the above theorem is the following property of the multivalued conditional expectation.

THEOREM 5.3. *Assume that* (Ω, Σ, μ) *is a positive, finite measure space,* Σ_0 *is a sub-σ-field of* Σ, *and X is a weakly sequentially complete separable Banach space. If* $F\colon \Omega \to P_{wkc}(X)$ *is integrably bounded then* $E^{\Sigma_0}F(\omega) \in P_{wkc}(X)$ μ-*α.e.*

PROOF. From Theorem 4.2 of [31], we know that S_F^1 is w-compact in $L^1(X)$. From the definition of the set valued conditional expectation (see §2) we have that $S^1_{E^{\Sigma_0}F} = E^{\Sigma_0}S_F^1 \in P_{wkc}(X)$ (note that $E^{\Sigma_0}\colon L^1(X) \to L^1(X, \Sigma_0)$ is continuous, linear and so is weakly continuous, hence maps w-compact sets into w-compact sets). Finally, apply Theorem 5.2 to conclude that $E^{\Sigma_0}F(\omega) \in P_{wkc}(X)$ μ-α.e. Q.E.D.

We can have a converse of Theorem 5.2 that also complements Theorem 4.2 of [31] (see also Proposition 3.1 of [32]).

THEOREM 5.4. *Assume that* (Ω, Σ, μ) *is a positive, finite measure space and X is a separable Banach space that contains no copy of* l^1. *If* $F\colon \Omega \to P_{fc}(X)$ *is integrably bounded then* S_F^1 *is a w-compact, convex subset of* $L^1(X)$.

PROOF. Convexity follows immediately from the convexity of the values of $F(\cdot)$. From Bourgain [1], we know that S_F^1 has no l^1-sequence and clearly is bounded. An appeal to Lemma I tells us that S_F^1 is relatively w-compact. But S_F^1 is closed and because of convexity is w-closed. So S_F^1 is w-compact in $L^1(X)$. Q.E.D.

We have a result analogous to Theorem 5.2 but with a different set of hypotheses.

THEOREM 5.5. *Assume that* (Ω, Σ, μ) *is a positive, finite, complete measure space and X is separable Banach space with* X^* *separable too. If* $K \subseteq L^1(X)$ *is decomposable, w-compact, convex and all vector measures* $m\colon \Sigma \to X$ *s.t.* $m(A) \in K(A) = \{\int_A f(\omega)\, d\mu(\omega)\colon f \in K\}$ *for all* $A \in \Sigma$ *admit a Radon-Nikodým derivative in* $L^1(X)$, *then there exists* $F\colon \Omega \to P_{wkc}(X)$ *integrably bounded s.t.* $K = S_F^1$.

PROOF. As before, from Hiai and Umegaki [18] we know that there exists $F\colon \Omega \to P_{fc}(X)$ integrably bounded s.t. $K = S_F^1$. Let $M\colon \Sigma \to 2^X \setminus \{\varnothing\}$ be the map defined by $M(A) = \{\int_A f(\omega)\, d\mu(\omega)\colon f \in K = S_F^1\}$. Since $K = S_F^1$ is by hypothesis w-compact in $L^1(X)$, for every $A \in \Sigma$ we have that $M(A) \in P_{wkc}(X)$. Furthermore in previous proofs we have seen that $M(\cdot)$ is a multimeasure, which is clearly of bounded variation and μ-continuous. Applying Theorem 3 of Costé [5] we deduce that there exists $\hat{F}\colon \Omega \to P_{wkc}(X)$ integrably bounded s.t. for all $A \in \Sigma$

$$M(A) = \int_A \hat{F}(\omega)\, d\mu(\omega) \Rightarrow \sigma(x^*, M(A)) = \sigma\left(x^*, \int_A \hat{F}\right)$$

$$= \int_A \sigma(x^*, \hat{F}(\omega))\, d\mu(\omega).$$

But we also have that $\sigma(x^*, M(A)) = \int_A \sigma(x^*, F(\omega))\, d\mu(\omega)$. Therefore for all $A \in \Sigma$ we have

$$\int_A \sigma(x^*, F(\omega))\, d\mu(\omega) = \int_A \sigma(x^*, \hat{F}(\omega))\, d\mu(\omega) \Rightarrow \sigma(x^*, F(\omega))$$

$$= \sigma(x^*, \hat{F}(\omega)) \quad \text{for all } \omega \in \Omega \setminus N_{x^*},\ \mu(N_{x^*}) = 0.$$

Let $\{x_n^*\}_{n\geq 1}$ be dense in X^*. Let $N_1 = \bigcup_{n\geq 1} N_{x_n^*}$. Then $\mu(N_1) = 0$. Observing that $\sigma(\cdot, F(\omega))$ and $\sigma(\cdot, \hat{F}(\omega))$ are both strongly continuous for $\omega \in \Omega \setminus N_2$, $\mu(N_2) = 0$ (since both multifunctions are integrably bounded), then through a classical density argument we conclude that $\sigma(x^*, F(\omega)) = \sigma(x^*, \hat{F}(\omega))$ for all $x^* \in X^*$ and all $\omega \in \Omega \setminus (N_1 \cup N_2)$, $\mu(N_1 \cup N_2) = 0$. Hence $F(\omega) = \hat{F}(\omega)$ μ-α.e. By redefining $F(\cdot)$ to be equal to $\hat{F}(\cdot)$ on the exceptional μ-null set $N_1 \cup N_2$ we get the conclusion of the theorem. Q.E.D.

REMARK. Again if X has the R.N.P., then our hypothesis on the measure selectors of $K(\cdot)$ is automatically satisfied.

The above theorem also produces a corresponding result for the set valued conditional expectation. The proof is the same as that of Theorem 5.3 (using this time Theorem 5.5 instead of Theorem 5.2) and so is omitted.

THEOREM 5.6. *Assume that (Ω, Σ, μ) is a positive, finite, complete measure space with Σ_0 a sub-σ-field of Σ and X a separable Banach space with X^* separable too. If $F\colon \Omega \to P_{wkc}(X)$ is integrably bounded and every vector measure $m(\cdot)$ for which we have $m(A) \in \int_A F(\omega)\, d\mu(\omega)$, $A \in \Sigma$, admits a Radon-Nikodým derivative in $L^1(X)$, then $E^{\Sigma_0}F(\omega) \in P_{wkc}(X)$ μ-α.e.*

We will close this section with a useful observation concerning the set valued conditional expectation.

THEOREM 5.7. *Assume that (Ω, Σ, μ) is a positive, finite, complete measure space with Σ_0 a complete sub-σ-field of Σ and X a separable Banach space with X^* separable too. If $F\colon \Omega \to P_f(X)$ is integrably bounded, then $\sigma(x^*, E^{\Sigma_0}F(\omega)) = E^{\Sigma_0}\sigma(x^*, F(\omega))$ for all $x^* \in X^*$ and $\omega \in \Omega \setminus N$, $\mu(N) = 0$.*

PROOF. From Theorem 5.4 of Hiai and Umegaki [**18**] we know that $\mathrm{cl} \int_A F = \mathrm{cl} \int_A E^{\Sigma_0}F$ for all $A \in \Sigma_0$. Then for every $x^* \in X^*$ and for every $A \in \Sigma_0$ we have

$$\int_A \sigma(x^*, F(\omega))\, d\mu(\omega) = \int_A \sigma(x^*, E^{\Sigma_0}F(\omega))\, d\mu(\omega)$$

$$\Rightarrow \int_A E^{\Sigma_0}\sigma(x^*, F(\omega))\, d\mu(\omega) = \int_A \sigma(x^*, E^{\Sigma_0}F(\omega))\, d\mu(\omega).$$

Recall that $(\omega, x^*) \to \sigma(x^*, F(\omega))$ is Σ-measurable in ω and continuous in x^*, and furthermore for every $x^* \in X^*$ is integrably bounded by $\|x^*\|\,|F(\omega)|$. Hence $(\omega, x^*) \to \sigma(x^*, F(\omega))$ is a Σ-quasi-integrable integrand in the sense of Thibault [**45**] and so we can apply Proposition 7 of that paper and get $E^{\Sigma_0}\sigma(x^*, F(\omega)) = \sigma(x^*, E^{\Sigma_0}F(\omega))$ for all $x^* \in X^*$ and all $\omega \in \Omega \setminus N$, $\mu(N) = 0$. Q.E.D.

6. A representation theorem. In this section we prove a representation theorem for additive set valued operators that appear often in problems of mathematical economics. This result partially generalizes Theorem 3.1 of the author [**31**], which was proved in the context of separable, reflexive Banach spaces. Also our theorem is a set valued extension of the results of Drewnowski and Orlicz [**10**], Friedman and Katz [**13**], and Hiai [**17**].

Let (Ω, Σ, μ) be a positive, finite measure space and X a Banach space. A map T: $L^1(X) \to 2^X \setminus \{\varnothing\}$ is said to be additive if $\overline{T(f+g)} = \overline{T(f) + T(g)}$ for all f, $g \in L^1(X)$ with $(\operatorname{supp} f \cap \operatorname{supp} g) = 0$ ($\operatorname{supp} f$ = support set of the function f) and $T(0) = \{0\}$. We say that $T(\cdot)$ is weakly continuous if for every $x^* \in X^*$, $f \to \sigma(x^*, T(f))$ is continuous from $L^1(X)$ into R. Note if $T(\cdot)$ has bounded values and is continuous in the Hausdorff metric, then $T(\cdot)$ is weakly continuous (follows easily from Theorem II-18 of Castaing and Valadier [**3**]). Furthermore if X is reflexive and $T(\cdot)$ has closed, convex values and is continuous in the Kuratowski-Mosco convergence of sets, then $T(\cdot)$ is weakly continuous (for details see Theorem 4.6 of [**38**]).

THEOREM 6.1. *Assume that (Ω, Σ, μ) is a positive, finite, complete measure space and X a separable Banach space. If T: $L^1(X) \to P_{wkc}(X)$ is additive, weakly continuous and $T(v) \subseteq \|v\|_1 G$ with $G \in P_{wkc}(X)$, then there exists F: $\Omega \times X \to P_{wkc}(X)$ s.t.*

(1) *for every $x \in X$, $\omega \to F(\omega, x)$ is measurable,*

(2) *for every $\omega \in \Omega$, $x \to F(\omega, x)$ is continuous from X into $P_{wkc}(X)$ with Hausdorff topology corresponding to $w(X, X^*)$,*

(3) $F(\omega, 0) = \{0\}$,

(4) $T(v) = \int_\Omega F(\omega, v(\omega))\, d\mu(\omega)$ *for all $v \in L^1(X)$.*

PROOF. Let $x^* \in X^*$ and consider the map $u(x^*)$: $L^1(X) \to R$ defined by $u(x^*)(f) = \sigma(x^*, T(f))$. Clearly $u(x^*)(\cdot)$ is additive and continuous. So Theorem 5.2 of Hiai [**17**] tells us that there exists $\varphi(x^*)(\cdot, x) \in L^1$. We claim $\varphi(x^*)(\cdot, x) \in L^\infty$. Suppose not. Then for every $n \geqslant 1$ we can find $A_n \in \Sigma$ s.t. $\mu(A_n) > 0$ and $|\varphi(x^*)(\omega, x)| > n$ for every $\omega \in A_n$. Take $f_n = \chi_{A_n} x \in L^1(X)$. Then we have $\sigma(x^*, T(f_n)) = \int_{A_n} \varphi(x^*)(\omega, x)\, d\mu(\omega) > n\mu(A_n)$. On the other hand $\sigma(x^*, T(f_n)) \leqslant \|f_n\|_1 \sigma(x^*, G) \leqslant \mu(\Omega)\|x\|\sigma(x^*, G) < \infty$, a contradiction. So $\varphi(x^*)(\cdot, x) \in L^\infty$. Let $\rho(\cdot)$ be a positive, linear lift on L^∞. Such a lift exists by Ionescu-Tulcea [**21**]. Set $\hat{\varphi}(x^*)(\cdot, x) = \rho(\varphi(x^*)(\cdot, x))$. Then for all $(\omega, x) \in \Omega \times X$ we have

$$\hat{\varphi}(x_1^* + x_2^*)(\omega, x) \leqslant \hat{\varphi}(x_1^*)(\omega, x) + \hat{\varphi}(x_2^*)(\omega, x)$$

and $\hat{\varphi}(\lambda x^*)(\omega, x) = \lambda \hat{\varphi}(x^*)(\omega, x)$ for all $\lambda > 0$.

Thus we get that $x^* \to \hat{\varphi}(x^*)(\omega, x)$ is sublinear. Also note that $|\hat{\varphi}(x^*)(\omega, x)| \leqslant \mu(\Omega)\|x\|\sigma(x^*, G)$ for all $\omega \in \Omega$. Hence $x^* \to \hat{\varphi}(x^*)(\omega, x)$ is $m(X^*, X)$-continuous. Thus there exists $F(\omega, x) \in P_{wkc}(X)$ s.t. $\hat{\varphi}(x^*)(\omega, x) = \sigma(x^*, F(\omega, x))$. This together with Theorem III-37 of Castaing and Valadier [**3**] implies that $\omega \to F(\omega, x)$ is measurable. Also for all $A \in \Sigma$, we have

$$\int_A \sigma(x^*, F(\omega, x))\, d\mu(\omega) \leqslant \mu(\Omega)\|x\|\sigma(x^*, G)$$

$$\Rightarrow \sigma(x^*, F(\omega, x)) \leqslant \mu(\Omega)\|x\|\sigma(x^*, G) \quad \mu\text{-}\alpha.\text{e.}$$

Note that the above exceptional μ-null set is independent of x and x^*, since both functions involved in the inequality are strongly continuous in x and $m(X^*, X)$-continuous in x^* and at the same time X is separable, while X^* is m-separable (see Castaing and Valadier [**3**, Lemma III-37]). So finally we have $F(\omega, x) \subseteq \|x\| \cdot G$

μ-α.e. and invoking Corollary II-22 of [3] we deduce that it is continuous from X with the strong topology into $P_{wkc}(X)$ with the Hausdorff uniformity corresponding to $w(X, X^*)$. Also for all $v(\cdot) \in L^1(X)$, $\sigma(x^*, T(v)) = \int_\Omega \sigma(x^*, F(\omega, v(\omega))\, d\mu(\omega)$ (see Hiai [17]). Note that $(\omega, x) \to \sigma(x^*, F(\omega, x))$ is measurable in ω, continuous in x, i.e. a Carathéodory function. So Lemma III-14 of Castaing and Valadier [3] tells us that $(\omega, x) \to \sigma(x^*, F(\omega, x))$ is jointly measurable, hence $\omega \to \sigma(x^*, F(\omega, v(\omega))$ is measurable. Therefore we can write that $\sigma(x^*, T(v)) = \sigma(x^*, \int_\Omega F(\omega, v(\omega)))$ and from Proposition 3.1 of [32] we conclude that $T(v) = \int_\Omega F(\omega, v(\omega))\, d\mu(\omega)$. Since $\hat{\varphi}(x^*)(\omega, 0) = 0$ for all $x^* \in X^*$ (see Hiai [17]), we have that $F(\omega, 0) = \{0\}$. Q.E.D.

REMARK. If X is finite dimensional, then $x \to F(\omega, x)$ is continuous in the Hausdorff metric and so $F(\cdot, \cdot)$ is $\Sigma \times B(X)$-measurable.

7. Transition multimeasures. In this section we conduct a detailed study of the properties of transition multimeasures, analogous to the one for simple multimeasures that appeared in [33]. Such objects are useful in mathematical economics in connection with infinite exchange economies with production (see Hildenbrand [19]) and in calculus of variations and relaxed control systems (see [47] and the references therein).

We will start with a Radon-Nikodým theorem for transition multimeasures. Recall that a measurable space (T, Z) is said to be separable if there exists $Z_0 \subseteq Z$ at most countable s.t. $\sigma(Z_0) = Z$ (see Hoffmann-Jørgensen [20]).

THEOREM 7.1. *Assume that (Ω, Σ, μ) is a positive, σ-finite, complete measure space, (T, Z, λ) is a positive, finite, separable measure space, and X is a separable Banach space. If L: $\Omega \times Z \to P_{kc}(X)$ is a transition multimeasure s.t. for all $\omega \in \Omega$ and all $A \in Z$, $L(\omega, A) \subseteq |L(\omega)|(A)G(\omega)$ μ-α.e. where $\omega \to |L(\omega)|(T)$ is integrable and G: $\Omega \to P_{kc}(X)$ is integrably bounded, then there exists F: $\Omega \times T \to P_{kc}(X)$ jointly measurable, integrably bounded on T and for all $\omega \in \Omega$ and all $A \in Z$*

$$L(\omega, A) = \int_A F(\omega, t)\, d\lambda(t).$$

PROOF. Let $\hat{X}$ be the separable Banach space postulated by the Rädstrom embedding theorem (see Theorem 3.6 of Hiai and Umegaki [18] and Klein and Thompson [25, Theorem 17.2.1]).

Then let R: $\Omega \to 2^{L^1(\hat{X})}$ be defined by

$$R(\omega) = \left\{ f \in L^1(\hat{X}) \colon L(\omega, A) = \int_A i^{-1}(\hat{f})(t)\, d\lambda(t),\ A \in Z \right\}$$

where $i(\cdot)$ is the isometric embedding of the metric space $(P_{kc}(X), h)$ into $L^1(\hat{X})$. From Corollary 5.3 of Hiai [16], we know that for all $\omega \in \Omega$, $R(\omega) \neq \varnothing$. Note that because Z is countably generated and $\hat{X}$ is separable, then $L^1(\hat{X})$ is separable. Let $\{A_n\}_{n \geqslant 1} \subseteq Z$ be the countable family generating Z and let $\{x_m^*\}_{m \geqslant 1}$ be dense in X^* for the $m(X^*, X)$-topology. Then we have

$$R(\omega) = \bigcap_{\substack{m \geqslant 1 \\ n \geqslant 1}} \left\{ \hat{f} \in L^1(\hat{X}) \colon \sigma(x_m^*, L(\omega, A_n)) = \sigma\left(x_m^*, \int_{A_n} i^{-1}(\hat{f}) \right) \right\}.$$

Since $i(\cdot)$ is an isometry on $P_{kc}(X)$ with the Hausdorff metric, we see that for all $n, m \geqslant 1$, $\varphi_{n,m}(\omega, \hat{f}) = \sigma(x_m^*, L(\omega, A_n)) - \sigma(x_m^*, \int_{A_n} i^{-1}(\hat{f}))$ is a Carathéodory function (i.e. measurable in ω, continuous in $\hat{f}$) and since as we saw above $L^1(\hat{X})$ is separable, we deduce that $\varphi(\cdot, \cdot)$ is jointly measurable (see for example Lemma III-14 of Castaing and Valadier [**3**]). Therefore we have

$$V_{n,m} = \{(\omega, \hat{f}) \in \Omega \times L^1(\hat{X}) \colon \varphi_{n,m}(\omega, \hat{f}) = 0\} \in \Sigma \times B(L^1(\hat{X}))$$
$$\Rightarrow \operatorname{Gr} R = \bigcap_{\substack{n \geqslant 1 \\ m \geqslant 1}} V_{n,m} \in \Sigma \times B(L^1(\hat{X})).$$

Apply Aumann's selection theorem (see Saint-Beuve [**40**, Theorem 3]) to find r: $\Omega \to L^1(\hat{X})$ measurable s.t. $r(\omega) \in R(\omega)$ for all $\omega \in \Omega$. From our hypotheses on $|L(\cdot)|(T)$ and $G(\cdot)$, we deduce that $\omega \to r(\omega)$ is μ-integrable. So we can apply Lemma 16 of Dunford and Schwartz [**12**, p. 196] and get that there exists an integrable function u: $\Omega \times T \to \hat{X}$ s.t. $r(\omega)(\cdot) = u(\omega, \cdot)$ μ-α.e. Set $F(\omega, t) = i^{-1}(u(\omega, t))$. Then for all $\omega \in \Omega$ and all $A \in \Sigma$ we have $L(\omega, A) = \int_A F(\omega, t)\, d\lambda(t)$ and F: $\Omega \times T \to P_{kc}(X)$ has the desired properties. Q.E.D.

The next theorem establishes the existence of transition selectors for certain transition multimeasures.

Let T be a Polish space (i.e. a complete, separable metric space) and X a Banach space. By $C_b(T)$ we will denote the space of bounded continuous functions of T and by $C_b(T) \otimes X$ the space of bounded continuous functions on T with values in a finite-dimensional subspace of X.

THEOREM 7.2. *Assume that* (Ω, Σ, μ) *is a positive, σ-finite, complete measure space,* $(T, B(T))$ *is a Polish space with its Borel σ-field, and* X *is a reflexive, separable Banach space. If* L: $\Omega \times Z \to P_{wkc}(X)$ *is a transition multimeasure of bounded variation and if* v: $\Omega \to X$ *is a measurable map s.t.* $v(\omega) \in L(\omega, C)$ *for some* $C \in B(T)$ *and all* $\omega \in \Omega$, *then there exists a transition selector* $m(\cdot, \cdot)$ *s.t.* $m(\omega, C) = v(\omega)$ *for all* $\omega \in \Omega$.

PROOF. Let $M_b(T, X)$ be the space of vector measures, with bounded variation defined on $(T, B(T))$. On this vector space consider the topology of simple convergence on $C_b(T) \otimes X$. Then consider the multifunction R: $\Omega \to M_b(T, X)$ defined by

$$R(\omega) = \{m \in M_b(T, X) \colon m(A) \in L(\omega, A),\ A \in Z \text{ and } m(T) = v(\omega)\}.$$

From Theorem 2.3 of [**16**] we know that $R(\omega) \neq \varnothing$ for all $\omega \in \Omega$. Note that since T is a Polish space, $B(T)$ is countably generated (i.e. separable) and so if $\{A_n\}_{n \geqslant 1}$ is the countable generating family and $\{x_k^*\}_{k \geqslant 1}$ is dense in X^*, then $R(\omega) = \{m \in M_b(T, X)$: $(x_k^*, m(A_n)) \leqslant \sigma(x_k^*, M(A_n))$, $n, k \geqslant 1$ and $m(T) = v(\omega)\}$. Let $\varphi_{n,k}$: $\Omega \times M_b(T, X) \to R$ and ψ: $\Omega \times M_b(T, X) \to X$ be defined by

$$\varphi_{n,k}(\omega, m) = \sigma(x_k^*, L(\omega, A_n)) - (x_k^*, m(A_n))$$

and

$$\psi(\omega, m) = m(T) - v(\omega).$$

Then we have

$$R(\omega) = \bigcap_{\substack{n\geqslant 1\\ k\geqslant 1}} \{(\omega, m) \in \Omega \times M_b(T, X): \varphi_{n,k}(\omega, m) \geqslant 0, \psi(\omega, m) = 0\}.$$

Since X_w (i.e. the Banach space X with the weak topology) is a Souslin space, it is a Lindelöf space (see Saint-Beuve [**40**, p. 113]). So we can apply Proposition 1.5 of Nougués-Saint-Beuve [**29**] and get that $m \to m(A)$ is measurable from $\Sigma(X) \cap M_b(T, X)$ into X_w, where $\Sigma(X)$ is the smallest σ-field on the space $M(T, X)$ = vector measures on T with values in X, which makes measurable the maps $m \to (x^*, m(A))$ for all $x^* \in X^*$ and all $A \in \Sigma$. Then it is easy to see from their definition that both $\varphi_{n,k}(\cdot, \cdot)$ and $\psi(\cdot, \cdot)$ are $\Sigma \times [\Sigma(X) \cap M_b(T, X)]$-measurable. Hence

$$\operatorname{Gr} R \in \Sigma \times [\Sigma(X) \cap M_b(T, X)].$$

But note that $M_b(T, X) = \bigcup_{n \geqslant 1} M_n(T, X)$, where $M_n(T, X) = \{m \in M_b(T, X): |m| \leqslant n\}$ and for each $n \geqslant 1$ the topology of simple convergence on $C_b(T) \otimes X$ coincides with the topology of simple convergence of $C_b(T, X)$. Hence for each $n \geqslant 1$, $M_n(T, X)$ is Souslin $\Rightarrow \bigcup_{n \geqslant 1} M_n(T, X) = M_b(T, X)$ is Souslin with the topology of simple convergence on $C_b(T) \otimes X$. Also from Theorem III-60 of Dellacherie and Meyer [**8**], we know that $M_b^+(T)$ with the topology of narrow convergence ("topologie etroite", see Definition III-54 of [**8**]) is Polish. So we can apply Proposition 2.4 of [**29**] and get that $\Sigma(X) \cap M_b(T, X) = B(M_b(T, X))$ = Borel σ-field of the Souslin space $M_b(T, X)$. Hence finally we have

$$\operatorname{Gr} R \in \Sigma \times B(M_b(T, X))$$

with $M_b(T, X)$ being a Souslin space with the topology of simple convergence on $C_b(T) \otimes X$. Apply Theorem 3 of Saint-Beuve [**40**] to get $r: \Omega \to M_b(T, X)$ measurable s.t. $r(\omega) \in R(\omega)$ for all $\omega \in \Omega$. So for all $\omega \in \Omega$ and all $A \in B(T)$ we have $r(\omega)(A) \in L(\omega, A)$ and $r(\omega)(T) = v(\omega)$. Finally note that for all $A \in B(T)$, $\omega \to r(\omega)(A)$ is the composition of $\omega \to r(\omega)$ and $r(\omega)(A)$. Then from the construction of $r(\cdot)$ and Proposition 1.5 of [**29**] we get that $\omega \to r(\omega)(A)$ is measurable. Therefore $r(\cdot, \cdot)$ is the desired transition selector. Q.E.D.

REMARK. Since $\omega \to L(\omega, C)$ is measurable by definition, the measurable function $v(\cdot)$ always exists.

Using this theorem we can have a set valued, infinite-dimensional generalization of the results of McKinney.

THEOREM 7.3. *Assume that* (Ω, Σ, μ), $(T, B(T))$, *and* X *are as in Theorem* 7.2. *If* $L: \Omega \times B(T) \to P_{fc}(X)$ *is a transition multimeasure s.t.* $L(\omega, A) \subseteq \mu(A)G(\omega)$ *for all* $A \in B(T)$ *and all* $\omega \in \Omega \setminus N$, $\mu(N) = 0$ *with* $G: \Omega \to P_{wkc}(X)$ *integrably bounded and if* $x \in \int_\Omega L(\omega, C)\, d\mu(\omega)$, *then there exists transition selector* $m(\cdot, \cdot)$ *s.t.* $x = \int_\Omega m(\omega, C)\, d\mu(\omega)$.

PROOF. Set $N(D) = \int_\Omega L(\omega, D)\, d\mu(\omega)$, $D \in B(T)$. From the corollary of Proposition 3.1 in [**32**] we know that $N(D) \in P_{wkc}(X)$. Also using Theorem 2.2 of Hiai and Umegaki [**18**], for all $x^* \in X^*$ we have that $\sigma(x^*, N(D)) = \int_\Omega \sigma(x^*, L(\omega, D))\, d\mu(\omega)$. Theorem 4.2 of McKinney [**27**] tells us that for all $x^* \in X^*$, $D \to \sigma(x^*, N(D))$ is a measure. Hence $N(\cdot)$ is a weak multimeasure and in fact a multimeasure since it is $P_{wkc}(X)$-valued.

From the the definition of the set valued integral we know that there exists $f_c(\cdot) \in S^1_{L(\cdot, C)}$ s.t. $x = \int_\Omega f_c(\omega)\, d\mu(\omega)$. Apply Theorem 7.2 to find m: $\Omega \times B(T) \to X$ a transition selector of $L(\cdot, \cdot)$ s.t. $m(\omega, C) = f_c(\omega)$ for all $\omega \in \Omega$. Then $x = \int_\Omega m(\omega, C)\, d\mu(\omega)$. Q.E.D.

REMARK. With minor technical modifications both Theorems 7.2 and 7.3 can be proved with X being a separable, dual Banach space. The details are left to the reader.

Next we show that the set valued integral of a transition multimeasure defines a multimeasure on $\Sigma \times Z$. This way we extend to a set valued setting a well-known result from vector valued measures.

THEOREM 7.4. *Assume that (Ω, Σ, μ) is a positive, finite, complete measure space, (T, Z, λ) is a positive, finite, separable measure space, and X is a separable Banach space with the R.N.P. If L: $\Omega \times Z \to P_{wkc}(X)$ is a transition multimeasure which is λ-continuous and if we define $N(K) = \int_\Omega L(\omega, K(\omega))\, d\mu(\omega)$ for $K \in \Sigma \times Z$, where $K(\omega) = \{t \in T: (\omega, t) \in K\}$, then $N(\cdot)$ is a multimeasure on $\Sigma \times Z$ and*

$$N(K) = \left\{ \int_\Omega m(\omega, K(\omega))\, d\mu(\omega): m(\cdot, \cdot) \in TS_L \right\}.$$

PROOF. Fix $x^* \in X^*$. Then $(\omega, A) \to \sigma(x^*, L(\omega, A))$ is a transition measure. From Proposition III-2-1 of Neveu **[48]** we know there exists a measure $\varphi(x^*)(\cdot)$ on $\Sigma \times Z$ s.t. for all $K \in \Sigma \times Z$ we have

$$\varphi(x^*)(K) = \int_\Omega \int_T \chi_{K(\omega)}(t) \sigma(x^*, L(\omega, dt))\, d\mu(\omega)$$
$$= \int_\Omega \sigma(x^*, L(\omega, K(\omega)))\, d\mu(\omega).$$

From Neveu **[48]** and Theorem III-37 of Castaing and Valadier **[3]**, we have that $\omega \to L(\omega, K(\omega))$ is measurable. So

$$\int_\Omega \sigma(x^*, L(\omega, K(\omega)))\, d\mu(\omega) = \sigma\left(x^*, \int_\Omega L(\omega, K(\omega))\right)$$
$$= \sigma(x^*, N(K)) \Rightarrow \varphi(x^*)(K) = \sigma(x^*, N(K)).$$

Thus $N(\cdot)$ is a multimeasure. Then Theorem 1 of Godet-Thobie **[14]** tells us that $N(K) = \{n(K): n \in S_N\}$. But clearly from the definition of N, $N \ll \mu \times \lambda$ and since X has the R.N.P. we can find $f(\cdot, \cdot) \in L^1(\Omega \times T, X)$ s.t. for all $B \in \Sigma$ and all $A \in Z$ we have $n(B \times A) = \int_B \int_A \{ f(\omega, t)\, d\lambda(t)\, d\mu(\omega)$. Set $m_1(\omega, A) = \int_A f(\omega, t)\, d\lambda(t)$. Then for all $B \in \Sigma$, $A \in Z$ we have

$$n(B \times A) = \int_B m_1(\omega, A)\, d\mu(\omega) \in N(B \times A) = \int_B L(\omega, A)\, d\mu(\omega)$$
$$\Rightarrow m_1(\omega, A) \in L(\omega, A) \quad \text{for all } \omega \in \Omega \setminus P_A,\ \mu(P_A) = 0.$$

But recall that Z is countably generated. So let $\{A_n\}_{n \geqslant 1} \subseteq Z$ be the generating family. Set $P = \bigcup_{n \geqslant 1} P_{A_n}$. Then $\mu(P) = 0$ and $m_1(\omega, A) \in L(\omega, A)$ for all $\omega \in \Omega \setminus P$ and all $A \in Z$. As in the proof of Theorem 5 of Godet-Thobie **[14]** we can find $u\colon \Omega \to M(T, X)$ measurable s.t. $u(\omega) \in S_{L(\omega,\cdot)}$ for all $\omega \in \Omega$. Then set

$$m(\omega, A) = \begin{cases} m_1(\omega, A) & \text{for all } \omega \in \Omega \setminus P, \\ u(\omega)(A) & \text{for all } \omega \in P. \end{cases}$$

Observe that $m(\cdot, \cdot) \in TS_L$ and $n(B \times A) = \int_B m(\omega, A)\, d\mu(\omega)$ for all $B \in \Sigma$, $A \in Z$. Since $n(\cdot, \cdot)$ extends uniquely on $\Sigma \times Z$, the theorem follows from Theorem 1 of [**14**]. Q.E.D.

An immediate consequence of this theorem is the following description of the vectors belonging in the range of $N(\cdot)$.

COROLLARY I. *Assume that the hypotheses of Theorem* 7.4 *hold. If* $x \in \int_B L(\omega, A)\, d\mu(\omega)$, *with* $B \in \Sigma$ *and* $A \in Z$, *then there exists* $m(\cdot, \cdot) \in TS_L$ *s.t.* $x = \int_B m(\omega, A)\, d\mu(\omega)$.

Another consequence of Theorem 7.4 is the following Radon-Nikodým type result.

COROLLARY II. *Assume that* (Ω, Σ, μ) *is a positive, finite, complete measure space,* (T, Z, λ) *is a positive, finite, separable measure space, and* X *is a separable Banach space with the R.N.P. and a separable dual* X^*. *If* $L\colon \Omega \times Z \to P_{wkc}(X)$ *is a transition multimeasure and for all* $A \in Z$, $L(\omega, A) \subseteq \lambda(A)G(\omega)$ μ-$\alpha.e.$ *where* G: $\to P_{wkc}(X)$ *is integrably bounded, then there exists* $F\colon \Omega \times T \to P_{wkc}(X)$ *integrably bounded s.t.* $N(K) = \int_K F(\omega, t)\, d(\mu \times \lambda)(\omega, t)$ *for all* $K \in \Sigma \times Z$.

PROOF. We have already seen in the proof of Theorem 7.4 that $N(\cdot)$ is a $P_{wkc}(X)$-valued multimeasure. Also note that $|N|(K) \leqslant \lambda(K) \int_\Omega |G(\omega)|\, d\mu(\omega) \Rightarrow$ $N(\cdot)$ is of bounded variation and $\mu \times \lambda$-continuous. Apply Theorem 3 of Costé [**5**] to get $F\colon \Omega \times T \to P_{wkc}(X)$ integrably bounded s.t. for all $K \in \Sigma \times Z$ we have $N(K) = \int_K F(\omega, t)\, d(\mu \times \lambda)(\omega, t)$. Q.E.D.

In the final result of this paper, we show that "integration" with respect to a transition multimeasure generates a new transition multimeasure.

THEOREM 7.5. *Assume that* (Ω, Σ, μ) *is a positive, finite, complete measure space,* $(T, B(T))$ *is a Polish space with its Borel* σ-*field, and* X *is a reflexive, separable Banach space. If* $L\colon \Omega \times B(T) \to P_{wkc}(X)$ *is a transition multimeasure which is* λ-*continuous, is of bounded variation, and* $|L(\omega)|(T) \leqslant \psi(\omega)$ μ-$\alpha.e.$ *with* $\psi(\cdot) \in L^\infty$, *and if* $f\colon \Omega \times T \to R_+$ *is a jointly measurable function s.t.* $t \to f(\omega, t)$ *is l.s.c.,* $\alpha(\omega) \leqslant \inf_{t \in T} f(\omega, t)$ μ-$\alpha.e.$ $\alpha(\cdot) \in L^1$ *and* $f(\omega, \cdot) \in L^\infty(T)$ *for all* $\omega \in \Omega$, *then there exists a transition multimeasure* $N\colon \Omega \times B(T) \to P_{wkc}(X)$ *s.t. for all* $x^* \in X^*$, *all* $\omega \in \Omega$, *and all* $A \in B(T)$ *we have* $\sigma(x^*, N(\omega, A)) = \int_A f(\omega, t)\sigma(x^*, L(\omega, dt))$. *Furthermore we can write that* $N(\omega, A) = \mathrm{cl}\{\int_A f(\omega, t) m(\omega, dt)\colon m \in TS_L\}$.

PROOF. Let $\varphi(\omega, A, x^*) = \int_A f(\omega, t)\sigma(x^*, L(\omega, dt))$. Clearly, for all $\omega \in \Omega$ and all $x^* \in X^*$, $A \to \varphi(\omega, A, x^*)$ is a signed measure. Also from Uryshon's metrization theorem (see Dugundji [**11**, p. 195]), we know that T, being a Polish space, is homeomorphic to as Borel subset of the Hilbert cube $H = [0,1]^N$. Identifying T with its image we will assume that it is such a subset of H. Then for all $\omega \in \Omega$ and all

$A \in B(H)$ set $\hat{\sigma}(x^*, L(\omega, A)) = \sigma(x^*, L(\omega, T \cap A))$. Since $B(T) = B(H) \cap T$, we see that $\hat{\sigma}(x^*, L(\cdot, \cdot))$ is a transition measure on $\Omega \times B(H)$.

Let $f_n(\cdot, \cdot)$ be R_+-valued Carathéodory integrands on $\Omega \times H$ s.t. for all $(\omega, t) \in \Omega \times T$ we have $f_n(\omega, t) \uparrow f(\omega, t)$ as $n \to \infty$. Such a family of integrands exists by Lemma 10 of Thibault [**45**]. Then consider the following

$$P_n(\omega, A, x^*) = \int_A [f_n(\omega, t) \wedge n] \hat{\sigma}(x^*, L(\omega, dt))$$
$$= \langle f_n(\omega, \cdot) \wedge n, \hat{\sigma}(x^*, L(\omega, \cdot)) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality brackets between $C(H)$ and $M_b(H)$. But since $f_n(\cdot, \cdot)$ is a Carathéodory function and H is compact, from Proposition 4.1 of [**37**] we know that $\omega \to f(\omega, \cdot) \wedge n$ is measurable from Ω into $C(H)$. Also from Proposition 1.3 of Nougues and Saint-Beuve [**29**] we know that $\omega \to \hat{\sigma}(x^*, L(\omega, \cdot))$ is measurable from Ω into $M_b(H) =$ Radon measures on H. Hence $\omega \to p_n(\omega, A, x^*)$ is measurable. Furthermore from the monotone convergence theorem we have

$$p_n(\omega, A, x^*) \uparrow \varphi(\omega, A, x^*) \quad \text{as } n \to \infty$$
$$\Rightarrow \omega \to \varphi(\omega, A, x^*) \quad \text{is measurable.}$$

Next we will show that for all $\omega \in \Omega$ and all $A \in B(T)$, $x^* \to \varphi(\omega, A, x^*)$ is sublinear. So let $\lambda \geqslant 0$. We have

$$\varphi(\omega, A, \lambda x^*) = \int_A f(\omega, t) \sigma(\lambda x^*, L(\omega, dt)).$$

But $\sigma(\lambda x^*, L(\omega, A)) = \lambda \sigma(x^*, L(\omega, A)) \Rightarrow \sigma(\lambda x^*, L(\omega, dt)) = \lambda \sigma(x^*, L(\omega, dt))$. Hence $\varphi(\omega, A, \lambda x^*) = \int_A \lambda f(\omega, t) \sigma(x^*, L(\omega, dt)) = \lambda \varphi(\omega, A, x^*) \Rightarrow x^* \to \varphi(\omega, A, x^*)$ is positively homogeneous. Also let $x_1^*, x_2^* \in X^*$. For all $A \in B(T)$ we have

$$\sigma(x_1^* + x_2^*, L(\omega, A)) \leqslant \sigma(x_1^*, L(\omega, A)) + \sigma(x_2^*, L(\omega, A))$$
$$\Rightarrow \sigma(x_1^* + x_2^*, L(\omega, \cdot)) \leqslant \sigma(x_1^*, L(\omega, \cdot)) + \sigma(x_2^*, L(\omega, \cdot))$$
$$\Rightarrow [\sigma(x_1^* + x_2^*, L(\omega, \cdot))]^+ \leqslant [\sigma(x_1^*, L(\omega, \cdot)) + \sigma(x_2^*, L(\omega, \cdot))]^+$$
$$\text{and } [\sigma(x_1^* + x_2^*, L(\omega, \cdot))]^- \geqslant [\sigma(x_1^*, L(\omega, \cdot)) + \sigma(x_2^*, L(\omega, \cdot))]^-.$$

Using the above inequalities and the fact that $f(\cdot, \cdot)$ is a positive integrand we can easily see that $\varphi(\omega, A, x_1^* + x_2^*) \leqslant \varphi(\omega, A, x_1^*) + \varphi(\omega, A, x_2^*) \Rightarrow \varphi(\omega, A, \cdot)$ is sublinear.

Next fix $x^* \in X^*$, $\omega \in \Omega$ and let $u(x^*)(\omega, \cdot) \in L^1(T)$ be the Radon-Nikodým derivative of $\sigma(x^*, L(\omega, \cdot))$ with respect to λ. We then have

$$\sigma(x^*, L(\omega, A)) = \int_A u(x^*)(\omega, t)\, d\lambda(t).$$

Let $x_n^* \xrightarrow{s} x^*$. Since $L(\omega, A)$ is bounded, $x^* \to \sigma(x^*, L(\omega, A))$ is continuous. So we have

$$\begin{aligned}
\sigma(x_n^*, L(\omega, A)) \to \sigma(x^*, L(\omega, A)) \quad \text{as } n \to \infty \\
&\Rightarrow \int_A u(x_n^*)(\omega, t)\, d\lambda(t) \to \int_A u(x^*)(\omega, t)\, d\lambda(t) \\
&\Rightarrow u(x_n^*)(\omega, \cdot) \to u(x^*)(\omega, \cdot) \quad \text{weakly in } L^1(T) \text{ for all } \omega \in \Omega \\
&\Rightarrow \int_A f(\omega, t) u(x_n^*)(\omega, t)\, d\lambda(t) \to \int_A f(\omega, t) u(x^*)(\omega, t)\, d\lambda(t) \\
&\Rightarrow \varphi(\omega, A, x_n^*) \to \varphi(\omega, A, x^*) \\
&\Rightarrow \varphi(\omega, A, \cdot) \quad \text{is strongly continuous.}
\end{aligned}$$

So we have seen that $x^* \to \varphi(\omega, A, x^*)$ is sublinear and strongly continuous. Because X is reflexive we can apply Theorem II-16 of Castaing and Valadier [3] and get $N\colon \Omega \times B(T) \to P_{wkc}(X)$ s.t. $\varphi(\omega, A, x^*) = \sigma(x^*, N(\omega, A))$. Since $\varphi(\cdot, \cdot, x^*)$ is a transition measure, using Theorem III-37 of [3] and Proposition 3 of Godet-Thobie [14] we conclude that $N(\cdot, \cdot)$ is a transition multimeasure.

Next let $x^* \in X^*$ and consider

$$L(x^*)(\omega, A) = \{z \in L(\omega, A)\colon (x^*, z) = \sigma(x^*, L(\omega, A))\}.$$

Note that $L(x^*)(\omega, A) \in P_{wkc}(X)$, and for all $x^* \in X^*$ and all $A \in B(T)$ we have $\operatorname{Gr} L(x^*)(\cdot, A) = \{(\omega, z) \in \Omega \times X\colon\ (x^*, z) - \sigma(x^*, L(\omega, A)) = 0\} \cap \operatorname{Gr} L(\cdot, A)$. Observe that $\omega \to \sigma(x^*, L(\omega, A))$ is measurable and so $(\omega, z) \to (x^*, z) - \sigma(x^*, L(\omega, A))$ is a Carathéodory function, hence jointly measurable. Also recall that $\operatorname{Gr} L(\cdot, A) \in \Sigma \times B(X)$. So finally $\operatorname{Gr} L(x^*)(\cdot, A) \in \Sigma \times B(X) \Rightarrow \omega \to L(x^*)(\omega, A)$ is measurable for all $x^* \in X^*$ and all $A \in B(T)$. Furthermore from Theorem 2.3 of Costé and Pallu de la Barrière [7] we know that $A \to L(x^*)(\omega, A)$ is a multimeasure. Hence we conclude that $L(x^*)(\cdot, \cdot)$ is a transition multimeasure. Using Theorem 5 of Godet-Thobie [14], we can find a transition selector $m(x^*)(\cdot, \cdot)$ s.t.

$$(1) \qquad \int_A f(\omega, t) \sigma(x^*, L(\omega, dt)) = \int_A f(\omega, t)(x^*, m(x^*)(\omega, dt)).$$

Also it is clear that

$$\begin{aligned}
(2) \qquad & \left\{\int_A f(\omega, t) m(\omega, dt)\colon m \in TS_L\right\} \subseteq N(\omega, A) \\
&\Rightarrow \overline{\operatorname{conv}}\left\{\int_A f(\omega, t) m(\omega, dt)\colon m \in TS_L\right\} \subseteq N(\omega, A).
\end{aligned}$$

Since $m(x^*)(\cdot, \cdot) \in TS_L$, from (1) and (2) above we deduce that $\overline{\operatorname{conv}}\{\int_A f(\omega, t) m(\omega, dt)\colon m \in TS_L\} = N(\omega, A)$.

Now we claim that the map $r(x^*)(m) = (x^*, \int_A f(\omega, t) m(\omega, dt))$ is continuous from $S_{L(\omega, \cdot)}$ with the topology of simple w-convergence into R. To see this, fix $\omega \in \Omega$ and work as follows. Let $s(\cdot)$ be an $L^\infty(T)$-countably valued function. We have

$$\int_A s(t)m(\omega, dt) = \sum_{k=1}^{\infty} q_k m(\omega, A \cap B_k),$$

where $q_k \in R_+$ and $B_k \in B(T)$, $k \geqslant 1$. So $m \to (x^*, \int_A s(t)m(\omega, dt))$ is continuous. Let $\{s_n(\cdot)\}_{n\geqslant 1} \subseteq L^\infty(T)$ s.t. $s_n(t) \to f(\omega, t)$ uniformly λ-α.e. (the existence of such a sequence is guaranteed by Corollary 3 of Diestel and Uhl [**9**, p. 42]). Then

$$\int_A |f(\omega, t) - s_n(t)| \, |(x^*, m(\omega, dt))| \leqslant \int_A |f(\omega, t) - s_n(t)| \sigma(x^*, \hat{L}(\omega, dt)),$$

where $\hat{L}(\omega, A) = \overline{\text{conv}}[L(\omega, A) \cup (-L(\omega, A))]$ for all $\omega \in \Omega$, $A \in B(T)$. Let $\hat{u}(x^*)(\omega, \cdot)$ be the Radon-Nikodým derivative of $\sigma(x^*, \hat{L}(\omega, \cdot))$. Then

$$\int_A |f(\omega, t) - s_n(t)| \, |(x^*, m(\omega, dt))|$$

$$\leqslant \int_A |f(\omega, t) - s_n(t)| \, |\hat{u}(x^*)(\omega, t)| \, d\lambda(t) \to 0$$

as $n \to \infty$ and clearly the convergence is uniform in $m(\omega, \cdot) \in S_{L(\cdot, \cdot)}$. Therefore

$$\varphi_n(m(\omega, \cdot)) = \int_A s_n(t)(x^*, m(\omega, dt))$$

$$\to \varphi(m(\omega, \cdot)) = \int_A f(\omega, t)(x^*, m(\omega, dt))$$

uniformly in $m(\omega, \cdot) \in S_{L(\omega, \cdot)} \Rightarrow \varphi(\cdot)$ is continuous on $S_{L(\omega, \cdot)}$ and so the claim follows.

From Theorem 1 of Godet-Thobie [**14**] we know that $S_{L(\omega, \cdot)}$ is compact in the topology of simple w-convergence. Combining this fact with the continuity of $\varphi(\cdot)$ proved above, we deduce that $\{\int_A f(\omega, t)m(\omega, dt) \colon m(\omega, \cdot) \in S_{L(\omega, \cdot)}\} \in P_{wkc}(X)$.

Let $R(\omega) = S_{L(\omega, \cdot)}$. Then Proposition 1.5 of Nougués-Saint-Beuve [**29**] tells us that $TS_L = S_R$. Also

$$\begin{aligned} \operatorname{Gr} R &= \{m \in M_b(T, X) \colon m(A) \in L(\omega, A), A \in B(T)\} \\ &= \{m \in M_b(T, X) \colon (x^*, m(A)) \leqslant \sigma(x^*, L(\omega, A)), x^* \in X^*, A \in B(T)\} \\ &= \bigcap_{n, k \geqslant 1} \{m \in M_b(T, X) \colon (x_k^*, m(A_n)) \leqslant \sigma(x_k^*, L(\omega, A_n))\}, \end{aligned}$$

where $\{x_k^*\}_{k\geqslant 1}$ is dense X^* and $\sigma(\{A_n\}_{n\geqslant 1}) = B(T)$. On $M_b(T, X)$ consider the topology of $C_b(T) \otimes X$-simple convergence. Then since $m \to (x_k^*, m(A_n))$ is continuous for the above topology for every n, $k \geqslant 1$, we get that $\operatorname{Gr} R \in \Sigma \times B(M_b(T, X))$. Recall that $M_b(T, X)$ with the above topology is a Souslin space (see Saint-Beuve [**41**]). Thus we can apply Theorem 5.10 of Wagner [**46**] and find r_n: $\Omega \to M_b(T, X)$, $n \geqslant 1$, measurable s.t.

$$R(\omega) = \text{cl}\{r_n(\omega)\}_{n\geqslant 1} \to N(\omega, A)$$

$$= \text{cl}\left\{\int_A f(\omega, t)m(\omega, dt) \colon m \in TS_L\right\}. \quad \text{Q.E.D.}$$

We can relax the reflexivity hypothesis on X, at the expense of introducing a stronger boundedness hypothesis on $L(\cdot, \cdot)$.

THEOREM 7.6. *Assume that* (Ω, Σ, μ) *and* $(T, B(T))$ *are as in Theorem* 7.5 *and that* X *is a separable Banach space. If* $L: \Omega \times B(T) \to P_{wkc}(X)$ *is a transition multimeasure of bounded variation, with* $L(\omega, A) \subseteq \lambda(A)G(\omega)$ μ*-a.e. for all* $A \in B(T)$ *and for* $G: \Omega \to P_{wkc}(X)$ *integrably bounded, and if* $f: \Omega \times T \to R_+$ *is as in Theorem* 7.5, *then there exists a transition multimeasure* $N: \Omega \times B(T) \to P_{wkc}(X)$ *s.t. for all* $x^* \in X^*$, *all* $\omega \in \Omega$, *and all* $A \in B(T)$ *we have*

$$\sigma(x^*, N(\omega, A)) = \int_A f(\omega, t)\sigma(x^*, L(\omega, dt)).$$

PROOF. The proof is the same as in Theorem 7.5. Only now note that

$$\left|\int_A f(\omega, t)\sigma(x^*, L(\omega, dt))\right| \leqslant \int_A |f(\omega, t)|\sigma(x^*, \hat{G}(\omega))\, d\lambda(t),$$

where $\hat{G}(\omega) = \overline{\text{conv}}[G(\omega) \cup (-G(\omega))]$. So $x^* \to \varphi(\omega, A, x^*)$ is $m(X^*, X)$-continuous and this implies the existence of $N(\omega, A) \in P_{wkc}(X)$ s.t. $\varphi(\omega, A, x^*) = \sigma(x^*, N(\omega, A))$. Finally recall that X^* is $m(X^*, X)$-separable (see Lemma III-37 of [**3**]). Q.E.D.

Applications of the theory of multifunctions and multimeasures can be found in [**22, 23, 34, 35**] (multifunctions) and [**36**] (multimeasures).

ACKNOWLEDGMENT. The author would like to express his deep gratitude to the referee for his very constructive criticism and comments which helped improve the presentation considerably.

REFERENCES

1. J. Bourgain, *An averaging result for l^1-sequences and applications to weakly conditionally compact sets in L^1_X*, Israel J. Math. **32** (1979), 289–299.

2. J. Brooks and N. Dinculeanu, *Weak compactness in spaces of Bochner integrable functions and applications*, Adv. in Math. **24** (1977), 172–188.

3. C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Math., vol. 580, Springer-Verlag, Berlin and New York, 1977.

4. L. Cesari, *Convexity of the range of certain integrals*, SIAM J. Control **13** (1975), 666–676.

5. A. Costé, *La propriété de Radon-Nikodym en intégration multivoque*, C. R. Acad. Sci. Paris **280** (1975), 1515–1518.

6. ______, *Sur les multimesures a valeurs fermées bornées s'un espace de Banach*, C. R. Acad. Sci. Paris **280** (1975), 567–570.

7. A. Costé and R. Pallu de la Barrière, *Radon-Nikodym theorems for set valued measures whose values are convex and closed*, Ann. Soc. Math. Polon. Series I: Comm. Math. **20** (1978), 283–309.

8. C. Dellacherie and A. Meyer, *Probabilities and potential*, Math. Studies, vol. 29, North-Holland, Amsterdam, 1978.

9. J. Diestel and J. Uhl, *Vector measures*, Math. Surveys, vol. 15, Amer. Math. Soc., Providence, R.I., 1977.

10. L. Drewnowski and W. Orlicz, *Continuity and representation of orthogonally additive functionals*, Bull. Acad. Polon. Sci. Math. **17** (1969), 647–653.

11. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966.

12. N. Dunford and J. Schwartz, *Linear operators*, Wiley, New York, 1958.

13. N. A. Friedman and M. Katz, *Additive functionals on L_p-spaces*, Canad. J. Math. **18** (1966), 1264–1271.

14. C. Godet-Thobie, *Some results about multimeasures and their selectors*, Measure Theory at Oberwolfach 1979 (Kolzow, ed.), Lecture Notes in Math., vol. 794, Springer-Verlag, Berlin, 1977, pp. 112–116.

15. ______, *Théorèmes de Radon-Nikodým multivoque et intégration par rapport à certaines multi-mesures*, Séminaire d'Analyse Convexe, Exposé no. 11, Montpellier, 1974.

16. F. Hiai, *Radon-Nikodym theorems for set valued measures*, J. Multivariate Anal. **8** (1978), 96–118.

17. ______, *Representation of additive functionals on vector valued normed Köthe spaces*, Kodai Math. J. **2** (1979), pp. 303–313.

18. F. Hiai and H. Umegaki, *Integrals, conditional expectations and martingales of multivalued functions*, J. Multivariate Anal. **7** (1977), 149–182.

19. W. Hildenbrand, *Core and equilibria of a large economy*, Princeton Univ. Press, Princeton, N. J., 1974.

20. J. Hoffman-Jørgensen, *The theory of analytic spaces*, Publ. no. 10, Aarhus Univ., 1970.

21. A. Ionescu-Tulcea and C. Ionescu-Tulcea, *Topics in the theory of lifting*, Springer-Verlag, Berlin, 1969.

22. M. A. Khan and N. S. Papageorgiou, *Cournot-Nash equilibria for generalized qualitative games with a continuum of players*, Nonlinear Anal. T.M.A. (to appear).

23. ______, *Cournot-Nash equilibria in generalized qualitative games with an atomless measure space of agents*, Proc. Amer. Math. Soc. **100** (1987), 505–510.

24. H. A. Klei, *Sous-ensembles de $L^1(E)$ sans suite-l^1*, C. R. Acad. Sci. Paris **295** (1982), 79–80.

25. E. Klein and A. Thompson, *Theory of correspondences*, Wiley, New York, 1985.

26. P.-J. Laurent, *Approximation and optimisation*, Hermann, Paris, 1972.

27. J. McKinney, *Kernels of measures on completely regular spaces*, Duke Math. J. **40** (1973), 915–923.

28. U. Mosco, *On the continuity of the Young-Fenchel transform*, J. Math. Anal. Appl. **35** (1971), 518–535.

29. M. F. Nouguès-Saint-Beuve, *Propriétés de mesurabilité dans les espaces de mesures*, Séminaire d'Analyse Convexe, Exposé no. 5, Montpellier, 1982.

30. R. Pallu de la Barrière, *Introduction a l'etude des multimesures*, Seminaire d'Initiation a l'Analyse (G. Choquet et. al., eds.), 19e année, no. 7, 1979/80.

31. N. S. Papageorgiou, *Representation of set valued operators*, Trans. Amer. Math. Soc. **292** (1985), 557–572.

32. ______, *On the theory of Banach space valued multifunctions. Part* 1: *Integration and conditional expectation*, J. Multivariate Anal. **17** (1985), 185–206.

33. ______, *On the theory of Banach space valued multifunctions. Part* 2: *Set valued martingales and set valued measures*, J. Multivariate Anal. **71** (1985), 207–227.

34. ______, *On the efficiency and optimality of random allocations*, J. Math. Anal. Appl. **105** (1985), 113–136.

35. ______, *On the efficiency and optimality of allocations*. II, SIAM J. Control Optim. **24** (1986), 452–479.

36. ______, *Efficiency and optimality in economies described by coalitions*, J. Math. Anal. Appl. **116** (1986), 497–512.

37. ______, *Random fixed point theorems for measurable multifunctions in Banach spaces*, Proc. Amer. Math. Soc. **97** (1986), 507–514.

38. ______, *Convergence theorems for Banach space valued integrable multifunctions*, Internat. J. Math. Math. Sci. (in press).

39. H. Rosenthal, *A characterization of Banach spaces containing l^1*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 2411–2413.

40. M.-F. Saint-Beuve, *On the extension of Von Neumann-Aumann's theorem*, J. Funct. Anal. **17** (1974), 112–129.

41. ______, *Some topological properties of vector measures of bounded variation and its applications*, Ann. Mat. Pura Appl. **66** (1978), 317–379.

42. G. Salinetti and R. Wets, *On the relation between two types of convergence of convex functions*, J. Math. Anal. Appl. **60** (1977), 211–226.

43. ______, *On the convergence of sequences of convex sets in finite dimensions*, SIAM Rev. **21** (1979), 18–33.

44. M. Talagrand, *Weak Cauchy sequences in $L^1(E)$*, Amer. J. Math. **106** (1984), 703–724.

45. L. Thibault, *Esperances conditionelles d'integrandes semi-continues*, Ann. Inst. H. Poincaré Ser. B **17** (1981), 337–350.

46. D. Wagner, *Survey of measurable selection theorems*, SIAM J. Control Optim. **15** (1977), 859–907.

47. J. Warga, *Optimal control of differential and functional equations*, Academic Press, New York, 1970.

48. J. Neveu, *Bases mathématique du calcut des probabilités*, Masson, Paris, 1964.

Department of Mathematics, University of California, Davis, California 95616

Current address: Division of Mathematics, School of Technology, University of Thessaloniki, Thessaloniki 540-06, Greece